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SOME PROPERTIES OF M-SUBHARMONIC FUNCTIONS

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Abstract: It is proved in the paper that if a function $f(z)$ is M-harmonic in a polydisk, then the function $f^2(z)$ is M-subharmonic. In addition, the case is $\tilde{\Delta} f=\tilde{\Delta} f^{\alpha}=0 \ (\alpha \neq 0, \alpha \neq 1, \alpha \in \mathbb{R})$ proved to $f(z)$ be a n-harmonic function.

Keywords: Holomorphic Function, Harmonic Function, Subharmonic Function, -Harmonic Function, -Subharmonic Function, Pluriharmonic Function, Plurisubharmonic Function, -Harmonic Function, -Subharmonic Function.

Introduction

Suppose an $B-$open unit ball in $\mathbb{C}^n$ with center at the origin Bn, a $\varphi_a(z)-$linear fractional biholomoric mapping of the ball $\left(B=\{z \in \mathbb{C}^n : |z|<1\}, \partial B=\{\omega \in \mathbb{C}^n : |\omega|=1\}\right)$ onto itself of the following form:

$$\varphi_a(z)=\frac{a-P_a(z)-\left(1-|a|^2\right)^{\frac{1}{2}}\left(z-P_a(z)\right)}{1-\langle z,a\rangle}.$$

At $a \in B$, $a \neq 0$, $\varphi_a(0)=a$, $\varphi_a(a)=0$, $\varphi_0(z)=-z$, where the angle brackets denote the Hermitian dot product of $\langle z,w\rangle=\sum_{i=1}^{n} z_i \bar{w}_i$, $|z|=\langle z,z\rangle^{\frac{1}{2}}$, and $P_a(z)=\frac{\langle z,a\rangle}{\langle a,a\rangle}a$, $a \neq 0$, $P_0(z)=0$. Obviously $\varphi_a(0)=a$, $\varphi_a(a)=0$.

If $G=\{z \in \mathbb{C}^n : \langle z,a\rangle \neq 1\}$, then φ_a holomorphically maps G to $\mathbb{C}^n$. It is clear that $G \supset \bar{B}$, since $|a|<1$.

The invariant Laplace operator $\tilde{\Delta}$ of functions on doubly smooth functions in B is defined as follows:

$$\tilde{\Delta} f(z) = \Delta(f \circ \varphi_z)(0)$$

where $z \in B$, $\Delta -$are the usual Laplacian [3: 54].

The operator $\tilde{\Delta}$ is called invariant because it commutes with automorphisms of the ball B in the following sense: $\tilde{\Delta}(f \circ \varphi) = (\tilde{\Delta} f) \circ \varphi$, where $f \in C^2(G)(G \subset B)$, and $\varphi -$is any biholomorphic automorphism of the ball B.

For an arbitrary $\lambda \in \mathbb{C}$, let us denote by X_λ the space of all functions $f \in C^2(B)$($f \in C^2\left(U^n\right)$), satisfying the equation

$$\tilde{\Delta} f(z) = \lambda \cdot f(z).$$

Case $\lambda = 0$ is the most interesting. We will call elements of the space X_0 M-harmonic functions.

Definition 1. Let B be the unit ball in $\mathbb{C}^n$. The function $f \in C^2(B)$ is called M-harmonic (M-subharmonic) if $\tilde{\Delta} f = 0$ $(\tilde{\Delta} f \geq 0)$, where $\Delta f = (1-|z|^2)(\Delta f - 4\sum_{i,j=1}^{n} z_i \bar{z}_j \frac{\partial^2 f}{\partial z_i \partial \bar{z}_j})$ is the invariant Laplacian in B [3: 55].

For the polycircle $U^n \subset \mathbb{C}^n$ a similar definition holds.

Definition 2. Let $U^n -$be the unit polydisk in $\mathbb{C}^n$. The function $f \in C^2\left(U^n\right)$ is called M-harmonic (M-subharmonic) in U^n if $\tilde{\Delta} f = 0$ $(\tilde{\Delta} f \geq 0)$, where $\Delta f = 2\sum_{j=1}^{n} \left(1-|z_j|^2\right)^2 \frac{\partial^2 f}{\partial z_j \partial \bar{z}_j}$ is the invariant Laplacian in U^n [4: 24].

Results and Discussion

Theorem 1. If a function f is M-harmonic in U^n, then f^2 is M-subharmonic in U^n.

Proof. For the function f^2, the invariant Laplacian has the following form:

$$\tilde{\Delta} f^2 = 2\sum_{j=1}^{n} \left(1-|z_j|^2\right)^2 \frac{\partial^2 f^2}{\partial z_j \partial \bar{z}_j} = 2\sum_{j=1}^{n} \left(1-|z_j|^2\right)^2 \left(2\frac{\partial f}{\partial z_j}\frac{\partial f}{\partial \bar{z}_j} + 2f\frac{\partial^2 f}{\partial z_j \partial \bar{z}_j}\right) =$$

$$= 4\sum_{j=1}^{n} \left(1-|z_j|^2\right)^2 \frac{\partial f}{\partial z_j}\frac{\partial f}{\partial \bar{z}_j} + 4f\sum_{j=1}^{n} \left(1-|z_j|^2\right)^2 \frac{\partial^2 f}{\partial z_j \partial \bar{z}_j} = 4\sum_{j=1}^{n} \left(1-|z_j|^2\right)^2 \frac{\partial f}{\partial z_j}\frac{\partial f}{\partial \bar{z}_j}.$$

By the conditions of the theorem, $\Delta f = 2\sum_{j=1}^{n} \left(1-|z_j|^2\right)^2 \frac{\partial^2 f}{\partial z_j \partial \bar{z}_j} = 0$ and if f is a real function, then $\frac{\partial f}{\partial \bar{z}_j} = \overline{\frac{\partial f}{\partial z_j}}$. From the following two equalities $\tilde{\Delta} f^2 = 4\sum_{j=1}^{n} \left(1-|z_j|^2\right)^2 \frac{\partial f}{\partial z_j}\frac{\partial f}{\partial \bar{z}_j} = 4\sum_{j=1}^{n} \left(1-|z_j|^2\right)^2 \left|\frac{\partial f}{\partial z_j}\right|^2$ it follows that for $\forall z_j \in B$

$(j=1,\ldots,n)$ with $\left(1-|z_j|^2\right)^2>0$ and $\left|\frac{\partial f}{\partial z_j}\right|^2\geq 0$, then $\tilde{\Delta} f^2\geq 0$. This means that the function f^2 is a M-subharmonic function.

Theorem 2. Suppose that $\tilde{\Delta} f=\tilde{\Delta} f^{\alpha}=0$ $(\alpha\neq 0,\alpha\neq 1,\alpha\in\mathbb{R})$ is in $U^n\subset\mathbb{C}^n$, then $f(z)$ is n-harmonic in U^n.

Proof. For the function f^{α}, the invariant of the Laplace operator has the form:

$$\tilde{\Delta} f^{\alpha}=2\sum_{j=1}^{n}\left(1-|z_j|^2\right)^2\frac{\partial^2 f^{\alpha}}{\partial z_j\partial\overline{z}_j}=2\sum_{j=1}^{n}\left(1-|z_j|^2\right)^2\left(\alpha(\alpha-1)f^{\alpha-2}\frac{\partial f}{\partial z_j}\frac{\partial f}{\partial\overline{z}_j}+\alpha f^{\alpha-1}\frac{\partial^2 f}{\partial z_j\partial\overline{z}_j}\right)=$$

$$=2\alpha(\alpha-1)f^{\alpha-2}\sum_{j=1}^{n}\left(1-|z_j|^2\right)^2\frac{\partial f}{\partial z_j}\frac{\partial f}{\partial\overline{z}_j}+2\alpha f^{\alpha-1}\sum_{j=1}^{n}\left(1-|z_j|^2\right)^2\frac{\partial^2 f}{\partial z_j\partial\overline{z}_j}=0.$$

From the conditions of the theorem $\Delta f=2\sum_{j=1}^{n}\left(1-|z_j|^2\right)^2\frac{\partial^2 f}{\partial z_j\partial\overline{z}_j}=0$ and $\alpha\neq 0$, $\alpha\neq 1$, $\alpha\in\mathbb{R}$ it follows that $\sum_{j=1}^{n}\left(1-|z_j|^2\right)^2\frac{\partial f}{\partial z_j}\frac{\partial f}{\partial\overline{z}_j}=0$. It is known that if f is a real function, then $\frac{\partial f}{\partial\overline{z}_j}=\overline{\frac{\partial f}{\partial z_j}}$. So,

$$\sum_{j=1}^{n}\left(1-|z_j|^2\right)^2\frac{\partial f}{\partial z_j}\frac{\partial f}{\partial\overline{z}_j}=\sum_{j=1}^{n}\left(1-|z_j|^2\right)^2\frac{\partial f}{\partial z_j}\overline{\frac{\partial f}{\partial z_j}}=\sum_{j=1}^{n}\left(1-|z_j|^2\right)^2\left|\frac{\partial f}{\partial z_j}\right|^2=0.$$

For $\forall z_j\in B$ $(j=1,\ldots,n)$. Then $\left|\frac{\partial f}{\partial z_j}\right|^2=\frac{\partial f}{\partial z_j}\frac{\partial f}{\partial\overline{z}_j}=0$. It follows that $\frac{\partial f}{\partial z_j}=0$ or $\frac{\partial f}{\partial\overline{z}_j}=0$. Then, $\frac{\partial^2 f}{\partial z_j\partial\overline{z}_j}=0$. This means that the function f is n-harmonic.

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